Competing Arguments for the Geometry Course: Why Were American High School Students Supposed to Study Geometry in the Twentieth Century?

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Abstract

This study contributes to the historical examination of the justification question for the particular case of the high school geometry course in the United States. The 20th century saw the emergence of competing arguments to justify the geometry course. Four modal arguments are identified including that geometry provides an opportunity for students to learn logic, that it helps develop mathematical intuition, that it affords students experiences that resemble the activity of the mathematician, and that it allows connections to the real world. Those arguments help understand what is put at stake by the various kinds of mathematical work that one can observe in contemporary geometry classes. The underlying assumptions of those arguments also help locate the influences in contemporary reform movements with regard to the study of geometry.

The geometry course has been a constant of the American high school curriculum throughout the 20th century but the arguments that justify it have been diverse. This paper explores the question “What are the stakes to be claimed by the study of geometry in the American high school?” We examine that question historically, looking at how the study of geometry in high school was justified during the 20th century. The geometry course is a particularly interesting case to examine different reasons for teaching mathematics because it was a locus of conflict, when, nearing the end of the 19th century, considerations of purpose, audience, and the emergence of new disciplines began to percolate the traditional humanist curriculum.

Herbert Kliebard (1995) has described the ensuing struggle for the American curriculum as the confrontation of various interest groups, which among other things, varied in the extent to which they supported the teaching of mathematics to every child. George Stanic (1986b) has argued that the sympathies that mathematics educators may have expressed toward one or another of those interest groups shaping the general curriculum at the beginning of the 20th
century may have been just superficial ways to channel the real debate on the mathematics curriculum: whether or not to fuse the study of the various mathematical disciplines (algebra, geometry, trigonometry) into an integrated course of mathematical studies. The integrated mathematics movement of the beginning of the 20th century made only a minor impact on the mathematics curriculum before being marginalized itself into a course for the less able (Stanic & Kilpatrick, 1992). The high school geometry course with its promise of training in mathematical reasoning was the beacon of non-integration. Marie Gugle, the first female president of the NCTM, said plainly “demonstrative geometry is logic and it will not fuse” (Gugle, 1926, p. 323).

The high school geometry course has survived in practice through the 20th century in spite of the arguments against its transfer value by educational psychologists and the arguments for integration with other mathematical domains by some mathematics educators. What is the value of the study of geometry that high school students can avowedly derive? In what way have the benefits expected from the study of geometry argued for the existence of a separate course of studies? To address those questions is of theoretical interest to us because the study of geometry in the US high school presents an interesting case where, in spite of a relatively well-defined mathematical domain, the influences on the curriculum as regards to that domain have been heterogeneous. An account of those justificatory arguments may help explain the diversity of practices that one can observe in the day-to-day work done in various versions of the geometry course in the United States, in spite of its apparent stability. Additionally, the current Standards movement in the US (NCTM, 1989, 2000) has advocated for stronger connections across mathematical strands of the curriculum. An answer to the question of what has been put at stake by the geometry course over the 20th century can help in understanding relevant issues in redesigning the geometry curriculum to respond to the demands for connections.

The High School Geometry Course

At the end of the 19th century, the Report from the Mathematics Conference of the Committee of Ten (Newcomb et al., 1893; Eliot et al., 1893/1969; Eliot, 1905) had argued the need for the geometry course on instrumental grounds: Being structured as an axiomatic-deductive body of knowledge, the study of geometry could educate the mental faculties of deductive reasoning and was therefore of value to all high school students (Halsted, 1893; Hill, 1895; Quast, 1968). The 20th century opened with the promise that geometry would achieve the goal of developing students’ capacities for deductive reasoning unlike any other subject, which would then transfer into reasoning capacity in other areas. The Report of the National Committee of Fifteen on the Geometry Syllabus, published in 1912, proved to be influential in successive years, especially in the writing of syllabi and textbooks (Quast, 1968). Despite the strong challenge to the notion of transfer issued by Thorndike’s research (Kilpatrick 1992; Stanic, 1986a; Thorndike, 1906, 1921, 1924; Thorndike & Woodworth, 1901), the geometry course has continued to exist as a main staple of the college preparatory curriculum.
At the turn of a new century, the publication of *Principles and Standards for School Mathematics (PSSM)* attempted to provide a new vision of the school mathematics curriculum. Rather than limiting the study of geometry to a particular course, *PSSM* establish new expectations for the teaching and learning of geometry across grade levels. According to *PSSM*, the study of geometry is meant to involve students in the experience of mathematical inquiry as well as make apparent to them how a mathematical domain changes over time.

By comparing the report of the Committee of Fifteen (Slaught et al., 1912) and *PSSM* (NCTM, 2000), a reader might find changes in the goals and outcomes of geometry instruction over the years. That contrast between expectations at the beginning and end of the 20th century serve as an initial illustration of our main point: Whereas the geometry course has endured across the 20th century, it has done so in spite of changing expectations.

**Methodological Considerations**

We produced this account through an analysis of historical documents that trace the path connecting the report of the Committee of Fifteen and *Principles and Standards*. These sources include professional articles on school geometry, curriculum documents, and geometry textbooks. They constitute an archive of the conventional wisdom regarding the geometry curriculum in the United States. We sampled key articles from this corpus, and examined those adapting some of the ideas that Michel Foucault developed for the historiography of ideas (Foucault, 1972). To analyze this corpus we grouped statements portraying similar views about the domain of school geometry or about the geometry student. We analyzed those groups of statements in search of the underlying views that answered the justification question of why students need to study geometry (Kliebard, 1977, 1982, 1995; Stanic, 1983/1984, 1986b).

We derived from that analysis a group of four arguments that have been used over the 20th century to justify the geometry course. We call these arguments “modal arguments” because they are lines of argument drawn upon by various pieces of text. The “modal arguments” are in that sense like the “ideal types” described by Max Weber (1949). We identified different arguments and the circumstances surrounding the birth of each of them by locating issues that each new argument seemed to address in providing a justification for the study of geometry.

An option in writing about the historical developments in the curriculum could have been to focus on key events, such as the meetings of committees, conferences, and boards. The report of the Committee of Fifteen is one such key event in the developments of the geometry curriculum. Other events during the 20th century have affected the geometry curriculum even though their avowed purpose was rarely focused on geometry. Our focus on the written record is an approximation to the historical development that circumvents the problem of determining the significance of any one of the events that punctuate the period of interest through examining the dispersion of ideas about the geometry course written in the period. As Foucault suggests, we consider those documents as
monuments—in need of description, comparison, and connection of similar kind as the one used by an archaeologist (Foucault, 1972, p. 7).

Our examination of the written record has concentrated on the *Mathematics Teacher* and we have used those records to trace others by following references. This journal, directed to an audience of mathematics teachers, and whose publication history spans the 20th century, is likely to refer us to the stakes of instruction in classrooms. Our inspection of that archival record could be complemented by inspections that use a similar approach but position themselves on different grounds. Our choice has been to reduce complexity in the number of sources in favor of keeping the amplitude of the period of interest wide.

**Branching Out: From the “Mental Discipline” Argument Towards a Manifold of Arguments for the Geometry Course**

The *Committee of Fifteen on the Geometry Syllabus* was formed in 1909 during the meeting of the National Education Association in Denver and in response to the request of the National Education Association and the American Federation of Teachers of the Mathematical and Natural Sciences in their respective meetings the previous year. The Committee worked for three years to write and publish a report that gave detailed information about expectations for the geometry course, fleshing out the more general recommendations from previous efforts by the Committee of Ten and the Committee on Mathematical Requirements (Eliot et al. 1893/1969; Nightingale et al., 1899).

Led by Herbert E. Slaught, a mathematics professor at the University of Chicago, the report started with a preview of the teaching of geometry in European countries and finished by providing a syllabus of the geometry course. Professional commitments among the members of the Committee of Fifteen varied; some were university professors, others were high school teachers in public or private, regular or technical high schools, and yet others were school administrators. Florian Cajori, a renowned historian of mathematics, wrote the historical review. Other notable members included William Betz, David Eugene Smith, and Eugene Randolph Smith.

The recommendations of the report of the Committee of Fifteen were understandably more specific than those of the Committee of Ten in providing a vision for the geometry course. For example, the report of the Committee of Ten had acknowledged students’ different abilities and interests but it had nevertheless suggested the same curriculum for all (Newcomb et al., 1893, p. 115-116). The uniformity in the syllabus constrained having different geometry courses for “various classes of students in the high schools” (Slaught et al., 1912, p. 89).

The report of the Committee of Fifteen suggested that students should spend less time solving original exercises. This suggestion defied the tendency of the late 19th century to engage students in original problems, which required students to use their reasoning to develop a novel proof rather than to replicate a proof they
had studied in the textbook (Herbst, 2002, p. 290). The members of the Committee of Fifteen argued that, “in accordance with the common practice of the past twenty years, this class of exercises has been magnified and extended, especially with reference to the more difficult exercises, beyond the interest and appreciation of the average pupil” (Slaught et al., 1912, p. 95). Instead, the Committee suggested, students could work on applications.

The Committee of Fifteen recognized that some geometrical notions had historically emerged from solutions to practical problems and showed examples of geometry problems in real world situations such as designing architectural elements, surveying, and sailing. The stress on applications included forging relationships between the topics included in the course and the skills for students to develop. For example, scale drawings were connected to the notion of similar triangles and labeled as “essential in surveying” (Slaught, et al., 1912, p. 120).

In various research articles published during the first quarter of the 20th century, Edward Thorndike had issued a challenge to the notion that a curriculum for every pupil should be designed on account of each discipline’s capacity to train the mental faculties. Thorndike’s studies attempted to show that transfer of training was not general but specific to the situations in which the training had occurred. In regard to its implications for the curriculum, he concluded,

> By any reasonable interpretation of the results, the intellectual values of studies should be determined largely by the special information, habits, interests, attitudes, and ideas which they demonstrably produce. The expectation of any large difference in general improvement of the mind from one study rather than another seems doomed to disappointment. (Thorndike, 1924, p. 98)

Thorndike and Woodworth published in 1901 the first results that put to question the curriculum theory of the humanists. But his progress on these issues certainly fed the discourse of interest groups that pushed to differentiate the curriculum according to students’ current needs or to students’ anticipated roles after their years of school education (Kliebard, 1995, p. 94).

In apparent acknowledgement of the challenge issued to the humanist curriculum by Thorndike’s work on transfer, the Committee of Fifteen took some distance from justifying the geometry course for its mental discipline value. Their stance on incorporating applied work as complementary to the teaching of rigorous proofs is justified by considering students’ abilities. This excerpt illustrates some of the underlying tensions between Eliot’s philosophy and Thorndike’s research.

> In the high school[,] geometry has long been taught because of its mind-training value only. This exclusive attention to the disciplinary side may be fascinating to mature minds, but in the case of young pupils it may lead to a dull formalism [,] which is unfortunate. On the other hand those who are advocating only a nominal amount of formal proof, devoting their time chiefly to industrial applications, are even more at
fault. The committee feels that a judicious fusion of theoretical and applied work, a fusion dictated by common sense and free from radicalism in either direction, is necessary. (Slaught, et al., 1912, p. 85)

Mental disciplinarians had held a “fundamentally optimistic view of human intelligence” (Stanic, 1986a, p. 40). Thus, from the perspective of mental disciplinarians all students were able to develop their abilities by having “access to the same knowledge” (ibid, p. 40). The report of the Committee of Fifteen proposed more of a balance between expectations for students to be taught rigorous proofs and applications of geometry. The combination of applied and theoretical work seems to have been the essential element distinguishing the report of the Committee of Fifteen from the report of the Committee of Ten. The report of the Committee of Ten had suggested that a major aim of demonstrative geometry was for students to discipline their logical reasoning faculties. Geometry, unlike algebra, was seen as an introduction to students of the “art of rigorous demonstration” (Newcomb et al., 1893, p. 115).

A main goal of learning of geometrical concepts in the report of the Committee of Fifteen was the development of reasoning skills that could transfer to other domains. Applications, while important, should not replace or decrease students’ exposure to formal proofs.

Certain writers on education have claimed that geometry has no distinctive disciplinary value, or that the formal side is so intangible that algebra and geometry should be fused into a single subject (not merely taught parallel to each other), which subject should occupy a single year and be purely utilitarian. These writers fail to recognize the fundamental significance of mathematics in either its intellectual or its material bearing. (Slaught, et al., 1912, p. 86)

Hence whereas the Committee of Fifteen encouraged attention to applications and the making of connections between algebra and geometry, it fundamentally endorsed a geometry course whose main commitment was with students’ development of reasoning skills. The syllabus included a set of theorems that provided the organization of topics to be taught, as a sort of backbone of the course, separating theorems that should be rigorously proved from theorems that could be presented informally. Thus, it seems as if making knowledge accessible to the learner preceded the need to preserve logic. Then, applications were to illustrate how geometric notions showed up in real life. Shibli (1932) argues that in doing this, the authors also took distance from the views of the Committee of Ten. The Committee of Fifteen did not expect students to just “be trained to draw inferences and follow short chains of reasoning” (p. 54).
The report of the Committee of Fifteen related to the issues raised by the Committee of Ten in different ways. On the one hand, as regards to the justification for the geometry course, the Committee of Fifteen did not ascribe to geometry a role in training students in the art of proving. While the report of the Committee of Ten justified the geometry course on a “mental discipline” argument, the report of the Committee of Fifteen did so on a “fusion” (Slaught, et al., 1912, p. 85) of applied and rigorous aspects of geometry. Trying to find common ground between applications of geometrical notions to the real world and formal aspects of geometry continued to be a theme in discussions about the geometry course in the future. On the other hand, as regards to the access question, the Committee of Fifteen endorsed the same principle that had been foundational for the Committee of Ten, namely that all students should have access to the same geometry course regardless of their career orientation or their ability.

**Four Modal Arguments for the Geometry Course**

Four “modal” arguments surfaced in the 20th century offering justification for the geometry course. By modal arguments we mean not necessarily ideologies explicitly promulgated by individuals but central tendencies around which the opinion of various individuals could converge. First, a *formal argument* defined the study of geometry as a case of logical reasoning (Christofferson, 1938; Fawcett, 1935, 1938, 1970; Meserve, 1962, 1972; Schlauch, 1930; Upton, 1930). Second, a *utilitarian argument* stated that geometry would provide tools for the future work or non-mathematical studies (Allendoerfer, 1969; Breslich, 1938). Third, a *mathematical argument* justified the study of geometry as an opportunity to experience the work of doing mathematics (Fehr, 1972, 1973; Henderson, 1940, 1947; Moise, 1975). Finally, an *intuitive argument* aligned the geometry course with opportunities to learn a language that would allow students to model the world (Betz, 1908, 1909, 1930; Cox, 1985; Hoffer, 1981; Usiskin, 1980; Usiskin & Coxford, 1972).

**A Formal Argument: Geometry Teaches to Use Logical Reasoning**

Proponents of a *formal argument* inherited the mental discipline argument and tried to fashion it in ways that could accommodate what was being learned about transfer. A major step from the publication of the Committee of Ten within those who favored the *formal argument* was the articulation of ways to enact the idea of teaching for transfer. The value of studying geometry was located in becoming skilled at building arguments, applying the same reasoning used in the geometry course. Proofs were not important because of the leverage they gave to understand particular mathematical concepts but because of the opportunity they created for students to learn, practice, and apply deduction. That is, geometric ideas were not as important as the method for making a logical argument. This method was said to be transferable to other domains such as newspaper reading and democratic participation.
In spite of Thorndike’s challenge to the notion of transfer of training, many of the mathematics educators who opposed the notion of an integrated mathematics curriculum and defended the need for every student to study mathematics advocated for the study of geometry on account of its formal training capabilities. Their argument was that if the transfer expected of geometry had not yet been shown, it was because the course had not been taught with transfer in mind, but that geometry could be taught for transfer. Harold Fawcett, who designed a geometry course that would teach geometry for transfer, said that

> If the real purpose of teaching demonstrative geometry is to give the pupil an understanding of the nature of proof, the emphasis should not be placed on the conclusions reached, but rather on the kind of thinking used in reaching these conclusions. (Fawcett, 1935, p. 466)

Fawcett’s seminal study of a geometry course taught for transfer became the 13th NCTM yearbook, *The Nature of Proof*. This book built on the formal argument by connecting the goals of geometry with the need for all students to learn the reasoning habits with which they could support and exercise the values of a democratic society (Fawcett, 1938, p. 75). Fawcett argued that learning how to do proofs in geometry is a skill needed by educated citizens because it prepares them for the task of analyzing a text logically and to reach conclusions. In another report, a group of mathematics educators that included Fawcett argued that

> Students should therefore learn geometry in order to learn to reason with equal rigor in other fields. Fundamentally the end sought is for the student to acquire both a thorough understanding of certain aspects of logical proof and such related attitudes and abilities as will encourage him to apply this understanding in a variety of life situations. (Bennett, et al., 1938, p. 188)

According to William Betz, who served as a president of NCTM (1932 -1934), the main goal of geometry was to combine experiences in the real world with abstract knowledge. In “The transfer of training, with particular reference to geometry,” Betz (1930) discussed the relevance of theories on transfer at the time. He concluded that teachers had an important role in helping students to experience the values of education in a democracy. Betz said that “geometry is a unique laboratory of thinking, and as such it fosters the persistent and systematic cultivation of the mental habits which are so essential to all those who would claim mental independence and genuine initiative as their birthright” (Betz, 1930, p. 194). He stressed that geometry is a special venue for the training of minds in developing tools that could be applied to other domains.

Bruce E. Meserve agreed with this perspective of teaching geometry for transfer. Meserve, who presided NCTM from 1964 to1966, proposed as the goal of teachers “to help each student develop his or her mathematical abilities (whether these abilities be very extensive or very limited) so that the student may have a greater potential for being an effective citizen in our modern society” (1962, p.
That is, regardless of students’ individual differences they should become productive members of society.

Ten years later, Meserve modified some of his recommendations including a program where informal geometry permeated all grade levels before a proof-based high school geometry course. Some of these changes included “to make increasing use of student explorations and conjectures, to stress logical concepts without becoming more formal, to welcome coordinate proofs and vector proofs as well as others, and to treat geometry as a part of mathematics” (1972, p. 181). While the new ideas suggest a move towards a justification centered on mathematics, Meserve continued to describe geometry as a case of logical reasoning.

Similarly, Halbert Christofferson who presided over NCTM from 1938 to 1940, argued that all students would be citizens required to reason and that geometry was essential for developing their reasoning skills. He provided many examples where the logical thinking of geometry was applied to the study of other kinds of propositions. Proofs were both a resource for developing thinking and a goal of the geometry course. Within his view, geometry “shows how thinking must be done if it is to be sound, dependable, rigorous” (Christofferson, 1938, p. 155).

Proponents of the formal argument stressed that the geometry course was the place to learn logical reasoning, unlike other courses in high school. W. S. Schlauch, honorary president of NCTM (1948-1953), argued that training in logical thinking was one of the main reasons for teaching demonstrative geometry. “In geometry more than in any other school subject, the learner is led to a belief in reason, and is made to feel the value of demonstration” (Schlauch, 1930, p. 134). Schlauch suggested to cover less theorems and to focus on discussing “numerous original exercises” (p. 142). While this expectation seems to be similar to the proponents of the mathematical argument, Schlauch’s emphasis on the development of formal mathematical thinking transferable to other domains exemplifies the views of those within the formal argument. His hope that students would engage in crafting original proofs seems to be a reminiscence of expectations in the report of the Committee of Ten and the practice of many, simple proof-exercises that became standard shortly thereafter (Herbst, 2002).

In sum, the main goal of the geometry course according to proponents of the formal argument was to have students learn to transfer skills and ways of thinking learned in geometry to other domains. None other high school mathematics course, this argument said, would carry on this responsibility as the geometry course.
A Utilitarian Argument: Geometry Prepares Students for the Workplace

A utilitarian argument was advanced to justify the geometry course on account of the need to prepare students for the needs of the workforce. This theme had its roots in the work of the Committee of Fifteen’s recommendation of applications of geometry, but it also resonates with the workplace preparation pitch of the social efficiency movement, one of the interest groups that Kliebard (1995) has identified as claiming a stake on the general curriculum debate of the 20th century. Within the utilitarian argument, decisions regarding the content of the geometry course were to be detached from any notion of mathematical activity. Rather, decisions as to what the geometry course should include were to be made according to the relevance of the topics in applying geometrical concepts or geometrical thinking to students' future occupations or professions or, as it became apparent during war times, to the needs of the country (Osborne & Crosswhite, 1970). While there is some overlap between the aims for the geometry course within the utilitarian argument and other arguments, such as the expectation for students to develop particular skills, to write proofs or to use their intuition, the expectation was to match students’ experiences in geometry with the demands of their future jobs.

Proponents of the utilitarian argument considered the geometry students as the future workers that they would become. For Ernst Rudolph Breslich, one of the aims of geometry was to provide tools for having educated citizens who would participate actively in the work force by applying practical notions of geometry. His suggestions for changes to the content of the geometry course were determined by the relevance of the topics in applying geometrical concepts or geometrical thinking to students' future occupations or professions. “Many adults firmly believe that in the training in reasoning and attacking problems in geometry they received something that was of definite value and help to them later in their occupations and professions” (Breslich, 1938, p. 312). Logical reasoning was one of the important elements of the geometry course within Breslich’s perspective. Yet, his emphasis on logic was different than among the proponents of the formal argument. Breslich (1938) focused on students’ use of these skills in their future jobs making constant references to future professions as opportunities to use geometric notions even when students may not show interest in knowing these applications.
The singularity of the *utilitarian argument* lies on the way these skills would be developed and the ultimate purpose for developing them. Rather than doing problems from the textbook, Breslich suggested to have students experience the work of professionals and to "train" students in acquiring skills as workers who are to do their job (Breslich, 1938, p. 311). Under the provisions of such utilitarian argument, Breslich stressed the process of using "the right kind of problems taken from life" (1938, p. 313). Thus, geometrical knowledge had the purpose of helping students to deal with everyday life.

In their chapter "Mathematics Education on the Defensive: 1920-1945," Alan Osborne and Joe Crosswhite (1970) document that some circumstances around the coming of the Second World War (e.g., that induction testing by the military showed evidence of incompetence in mathematics) came into the mathematics education rhetoric as arguments to teach "mathematical content with military uses" (p. 231). They also indicate, "concern for the mathematical competence of American youth extended beyond military needs to encompass the employment and training problems of increasingly technical industries" (p. 231). In particular, Euclidean geometry was then accused of being too abstract and, at the same time, recommendations were made to teach students methods of indirect measurement and basic principles of engineering and military work (Osborne & Crosswhite, 1970, pp. 232-233). The high school geometry course was under pressure to accommodate more practice in the study of formulas associated with measures of plane and solid figures and their applications, and to reduce the role of proof.

Another proponent of the utilitarian argument, Carl B. Allendoerfer, a mathematician at the University of Washington and former President of the MAA (1959-60), stated that making connections between geometry and other subject matters, especially science or technical careers, was of utmost importance. His suggestions about the content of the geometry course were influenced by applications of geometrical skills to other domains, as it is the case for including solid geometry. The means for getting students acquainted with geometric concepts were also related to applications. He stressed the notion that geometry ought to be taught in agreement with methods and goals of the workforce, by forging connections with teachers of vocational areas (Allendoerfer, 1969, p. 168). However, different from Breslich, Allendoerfer wanted the course to remain within mathematical activity when he said, "We must strive to teach our geometry courses with a truly geometric flavor, and not merely as an exercise in algebra or in logic" (Allendoerfer, 1969, p. 169). His emphasis on having a "formal deductive" (Allendoerfer, 1969, p. 169) course on plane geometry distinguished him from Breslich. Starting with informal experiences with geometry in earlier grades, students should have opportunities to have a formal geometry course. The ultimate goal would be to "apply our geometry to algebra, calculus, science, art architecture, and elsewhere" (Allendoerfer, 1969, p. 169).
The place of proof in the utilitarian argument is problematic. Breslich argued that the emphasis on producing proofs limits students' ability to engage in creative work and excludes many capable students. Though he accepted that deductive proofs provide reasons for why something is true, he insisted that they should follow an informal understanding of the geometrical notions. Similarly, Allendoerfer stated that good teachers should "not bury the geometry under an avalanche of rigor" (1969, p. 167). His worries about students seemed to take precedence over the norms or values from the discipline. While proponents of the utilitarian argument might argue in favor of including proofs within the geometry course, their concerns with preparing students for their future careers tended to be stronger than other aims.

A Mathematical Argument: Geometry for the Experience and the Ideas of Mathematicians

Within proponents of the mathematical argument, the geometry course had as a major goal that of having students experience the activity of mathematicians. Ways of attaining this goal varied. Some proponents argued that Euclidean geometry is an optimal context for students to engage in making and proving conjectures (Henderson, 1947; Moise, 1975). Others claimed that if students needed to follow the work of mathematicians they ought to give preeminence to non-Euclidean geometries by modeling the developments in the discipline (Fehr, 1972). One common notion among proponents of the mathematical argument was that the study of geometry remained within the realm of mathematical activity and focused on knowing geometry. Kenneth B. Henderson, who had authored a geometry textbook, expected students “to discover and test possible courses of action” (1947, p. 177) when they worked on a geometry problem, just like mathematicians do. Henderson highlighted distinctions between the work of mathematicians and that of empirical scientists. “The difference is that the scientist relies chiefly on experimental corroboration while the mathematician demonstrates the theorem as a necessary consequence of other theorems, postulates, or definitions” (1947, p. 177).

Henderson also stressed the importance of public discussion in the development of postulates and theorems in the course. Within his view, classroom discussion was so important that what textbooks prescribed had to be subordinated to the geometrical notions developed in class. Debates among students as they tried to produce convincing arguments were essential in learning geometry “as it is made rather than by imitation of the ‘canned’ proofs of the textbook” (Henderson, 1947, p. 177). That is, students should go beyond getting trained to write proofs, by encountering the work of proving in the context in which ideas emerged.

Some of the ideas that contributed to the mathematical argument preceded the publication of the report of the Committee of Fifteen. Indeed, Henderson’s views have strong resonances with those of Eugene Randolph Smith—a geometry teacher who had written a textbook that developed geometry by “the syllabus method.” In the preface of his book he had stated “the hope of encouraging
teachers to undertake Geometry by the ‘no text method’” (1909, p. 3). Smith’s book gave a list of definitions, axioms and theorems along with a variety of exercises including proofs, numerical problems and constructions. Smith’s suggestions were slightly different from those later proposed by the Committee of Fifteen (which he was part of) and which proposed to keep a close connection between theorems and relevant exercises.

Edwin Moise, a Harvard mathematics professor who co-authored a geometry textbook (Moise & Downs, 1964), expected students to engage in mathematical activity through problem solving. Within Moise’s view, students needed to face mathematics as a creative activity. Writing proofs on their own was an extension of the work done in class, the real test for understanding and the rite of passage for becoming a mathematician. “When students solve such problems—and they do—the gap between theory and homework vanishes. On these occasions the student is, probably for the first time in his life, working in his capacity as a mathematician” (Moise, 1975, p. 477). Henderson and Moise agreed upon setting the aims and the contents of the course within the realm of mathematical activity. Proofs became an important resource for students to understand geometric notions and more than a mere exercise on logic.

Moise was particularly interested in order and coherence. His proposed geometry course was mostly defined by its structure. The historical development of geometry was important in supporting his view of the geometry course as a year-long high school course and not integrated with other mathematics courses. “If the facts of elementary geometry were taught piecemeal, as digressions in other courses, with no regard to the way in which they fit together, then the educational effect would be quite different” (Moise, 1975, p. 477). Moise saw geometry as a distinct body of knowledge and thus stressed the importance of a unified geometry course.

Two other proposals are also characteristic of the mathematical argument: the proposal to integrate geometry with other courses, eliminating the one-year high school course on geometry and the proposal to use non-Euclidean geometries as the basis for high school geometry. Howard F. Fehr, who presided NCTM from 1956 to 1958, turned to changes in the discipline.

The survival of Euclid’s geometry rests on the assumption that it is the only subject available at the secondary school level to introduce students to an axiomatic development of mathematics. This was true a century ago. But recent advances in algebra, probability theory, and analysis have made it possible to use these topics in an elementary and simple manner, to introduce axiomatic structure. In fact, geometrical thinking today is vastly different from that used in the narrow synthetic approach. (Fehr, 1972, p. 151)

According to Fehr, high school geometry should model the work of current mathematicians. He defined geometry as it is connected to other branches of mathematics. Following Dieudonné, Fehr declared, “Mathematics is no longer conceived of as a set of disjoint branches, each evolving in its own way” (1972, p.
Consequently, preparing students for further studies in mathematics and related areas was of utmost importance. Students of geometry could be introduced to an axiomatic system within the framework of linear algebra. Fehr also worried about the isolation of geometry within high school mathematics as an American phenomenon.

Of all the developed countries of the world, the only country that retains a year sequence of a modified study of Euclid’s synthetic geometry is the United States. We must immediately give serious consideration to presenting our high school youth with a mathematical education that will not leave them anachronistic when they enter the university or enter the life of adult society. (Fehr, 1973, p. 379)

Thus, two different stances define the place of geometries other than Euclid’s synthetic geometry within the proponents of the mathematical argument. One stance uses non-Euclidean geometries as a place for students to understand how to deal with assumptions in a mathematical system. The other stance incorporates other geometries as a way to align the work in high school with the current work of mathematicians. While the range of possibilities in terms of the resources available for students and the definition of mathematical activity vary, these two stances share the aim of making geometry the place of experiencing the ideas and the work of mathematicians.

An Intuitive Argument: Geometric Expression Helps Students Interpret their Experiences in the World

The interplay between geometry and intuition permeates the justifications of the geometry course among different arguments. However, proponents of what we call the intuitive argument made a case for geometry as a unique opportunity for students to apply the intuition of the geometric objects to describing the world. This argument can be traced back to John Dewey’s (1903) views on the psychological and the logical in the teaching of geometry. There are variations among this argument regarding students’ engagement in mathematical activity. Some proponents responded to the need to develop students’ basic skills (e.g., calculating perimeter and area of figures) and thus call for developing geometric literacy (Cox, 1985; Hoffer, 1981). Others tended to go deeper in advocating that the course present alternative mathematical ideas that would be more aligned with students’ needs (Usiskin, 1980/1995; Usiskin & Coxford, 1972). The core idea sustaining proponents of the intuitive argument was the principle that geometry provides lenses to understand, to experience, and to model the physical world by forging stronger connections between experiences, intuitions, skills, and geometrical notions. Mathematics, as a human activity, allows bonding with the physical world through studying the spatial features of physical objects. Unlike other branches of mathematics, geometry was said to merge empirical knowledge about physical objects and abstract ways of dealing with those objects.
Proponents of the intuitive argument juggled tensions between the need for all students to acquire geometric literacy and differences in students’ abilities. For some, informal geometry appeared as a solution that would promote students’ interest in proving some conjectures formally later rather than starting with a formal treatment of the subject (Cox, 1985; Hoffer, 1981; Peterson, 1973). According to Peterson, “The use of informal geometry in what is usually considered a formal geometry course should make the study of geometry more interesting” (Peterson, 1973, p. 90). Thus, students’ motivation shaped decisions about the geometry course.

Philip Cox argued that, “No longer can geometry be considered an appropriate subject for study only by those with a special aptitude for mathematics” (1985, p. 404). According to Cox, the first semester of the geometry course should be devoted to studying the concepts informally, making the study of geometry more inclusive. “More informal geometry and informal geometry courses at the high school level are needed if we wish to have most of our students achieve some degree of geometric literacy” (Cox, 1985, p. 405). Cox also suggested various versions of the geometry course, tailored to different populations such as college-bound students and those who would pursue other careers and he wrote a textbook that illustrated the approach (Cox, 1992). In contrast with the formal argument, which intended to prepare educated citizens, and the mathematical argument, which viewed all students as budding mathematicians, the intuitive argument intended to cater different courses according to students’ intended needs.

Some proponents of the intuitive argument turned to the van Hiele levels of geometry learning (see Fuys, Geddes & Tischler, 1988) for deciding the range of skills that students ought to develop (Hoffer, 1981). The van Hiele levels categorize students’ experiences with geometry according to the kind of reasoning invested. Based on his experience teaching high school geometry, Hoffer recommended a transition from informal to formal geometry. Students’ work with proofs ought to happen after more informal experiences. Within his view, proofs were as important in the geometry course as other experiences that might not include proofs. He concluded, “geometry is more than proof” (1981, p. 18), in contrast with the mathematical argument for which proofs are essential in the construction of geometric ideas.
The informal geometry course has been adopted in various schools as an alternative for students’ mathematics curriculum requirements, especially for students in a low track. One of the characteristics of this course is that it minimizes and even eliminates proofs, substituting them by explorations. Usually, the informal geometry course would emphasize algebraic skills, using geometric properties as a context for reviewing or learning applications of Algebra I such as solving equations (see Hoffer & Koss, 1996).

A different perspective within the proponents of the intuitive argument has been supported by a mathematical examination of geometry. Zalman Usiskin has argued that the teaching of geometry should not focus solely on students being exposed to a mathematical system but should allow students to make connections between geometry and the real world. These connections lay beyond the development of particular skills and stress on the power of geometry to model real-life phenomena. For example, in their geometry textbook, Zalman Usiskin and Arthur Coxford (1972) chose a transformation approach to the geometry course because of possible connections with relevant mathematical ideas in other courses and in response to current ways of working with geometry (Coxford, 1973). This justification is closely related to the mathematical argument. At the same time, the opportunity to relate the high school geometry course with “previous intuitive ideas” (Usiskin & Coxford, 1972, p. 21) brings to the fore a geometry course that takes into account students’ intuition.

Usiskin’s suggestions about the geometry course foreshadowed the first publication of the NCTM Standards in 1989. Usiskin stated as reasons for teaching high school geometry that: “1. Geometry uniquely connects mathematics with the real physical world. 2. Geometry uniquely enables ideas from other areas of mathematics to be pictured. 3. Geometry non-uniquely provides an example of a mathematical system” (1980, p. 418). While the justifications for high school geometry had usually focused on the last reason, Usiskin argued that geometry had to provide opportunities for students to make connections with the real world.

From the quotes that accompany our discussion of each of the arguments, it is apparent that individual authors rarely subscribed to a unique, well-defined modal argument. Still, their writings permit to isolate those four modal arguments as ideal types of justifications for the study of geometry. Table 1 shows some of the essential elements characterizing each modal argument.
Table 1. Elements within the four modal arguments justifying the geometry course.

<table>
<thead>
<tr>
<th>Goals of the geometry course of studies</th>
<th>Formal argument</th>
<th>Utilitarian argument</th>
<th>Mathematical argument</th>
<th>Intuitive argument</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geometry is a case of logical reasoning.</td>
<td>Geometry is a tool for dealing with applications in other fields.</td>
<td>Geometry is a conceptual domain that permits to experience the work of mathematicians.</td>
<td>Geometry provides a language for our experiences with the real world.</td>
<td></td>
</tr>
<tr>
<td>Views about mathematical activity</td>
<td>Transferring formal geometry reasoning to logical abilities.</td>
<td>Studying concepts and problems that apply to work settings.</td>
<td>Applying deductive reasoning through the study of geometric concepts.</td>
<td>Modeling problems using geometric ideas while reasoning intuitively.</td>
</tr>
<tr>
<td>Expectations about students</td>
<td>All students require logical reasoning to be good citizens and to participate in a democracy.</td>
<td>All students will be part of the workforce in the future.</td>
<td>All students can simulate the work of mathematicians.</td>
<td>All students could develop skills but their abilities vary</td>
</tr>
<tr>
<td>Characteristics of problems in the geometry curriculum</td>
<td>Applying logical thinking to mathematical and real-life situations.</td>
<td>Relating geometric concepts and formulas to model real-world objects or to solve problems emerging in job situations.</td>
<td>Making conjectures and proving theorems deductively.</td>
<td>Exploring intuitively geometric ideas towards formality. Integrating algebra and geometry.</td>
</tr>
<tr>
<td>The place of proofs</td>
<td>Proofs give opportunities to practice deductive reasoning detached from geometric concepts.</td>
<td>Proofs are not as important as problems that apply geometry to future jobs.</td>
<td>Proofs of original problems provide opportunities to experience the activity of mathematicians.</td>
<td>Proofs follow informal appreciation of geometric concepts, blurring differences between definitions, postulates and theorems.</td>
</tr>
</tbody>
</table>
Principles and Standards as a Space for Convergence

Similar to the movement that proposed an integrated mathematics curriculum at the beginning of the 20th century, the recommendations of the Standards movement of the end of the 20th century include a strong impetus to connect mathematical domains, connect mathematical ideas with those of other disciplines, and connect mathematical ideas with problems from the real world (NCTM, 2000, pp. 64-66). PSSM does not provide a syllabus for any of the mathematics courses and does not directly disown a separated geometry course. Rather, it suggests that elements of geometry should permeate the school mathematics curriculum and proposes forging stronger connections between subjects in the event that separate courses be taught (NCTM, 2000, p. 289; see also pp. 354-359). The existence of a Geometry Standard among the five main content Standards confirms that students’ development of geometric knowledge is still valued. One of the consequences of a call for connections is that the justifications for the teaching and learning of geometric concepts permeate the mathematics curriculum without solely allocating that responsibility to any one course. In addition to that, the five process Standards (Problem Solving, Reasoning and Proof, Communications, Connections, and Representations) are all connected to geometric content.

Some of the modal arguments developed through the 20th century to justify the geometry course have percolated into the Standards. Justifications for the study of geometry retain some of the special properties that sustained its presence within the high school mathematics curriculum. At the same time, other justifications have faded from the high school mathematics curriculum. The way in which the aims of learning and teaching geometry manifest as well as the leverage of these goals in the mathematics curriculum have changed.

The 20th century opened with issues about transfer of mental discipline as justifications for the study of geometry. The entrenched notion among mathematics educators that the main role of geometry was to train in logic was arguably one culprit for the eventual failure of attempts at unification started at the beginning of the century with the general mathematics movement of E. H. Moore (1926/1902). A notable change in the rhetoric of the Standards movement is that in spite of the value put on students’ learning of geometry, the formal argument plays no role in the justification of the study of geometry within the rhetoric of the Standards movement. The Reasoning and Proof Standard embeds the justifications for the teaching of proof at all levels. But in contrast with the approach shown in Fawcett’s The Nature of Proof that provided examples of using deductive reasoning in non-mathematical situations, PSSM includes examples that lie within the realm of mathematical activity (or that relate to mathematical models of applied problems). Mathematical reasoning is not directly exported from geometry into non-mathematical situations; rather, the role of proof in creating reasonable mathematics is emphasized. For example, the Geometry Standard establishes that students should “put together a number of logical deductions” (NCTM, 2000, p. 310) when solving a problem. Students’ learning of logical reasoning is a tool for them to use in their mathematics classrooms,
especially when crafting proofs (NCTM, 2000, p. 342). But geometry does not carry the burden of teaching reasoning skills. Rather, students’ use of logical deductions (in mathematics) should lead students to have a deeper understanding of geometric notions (NCTM, 2000, p. 311), a goal that seems to be more aligned with the mathematical argument.

Whereas the formal argument emphasized the use of logical reasoning in situations outside of mathematics, PSSM highlights that students will be more empowered and autonomous in their pursuit of mathematical knowledge. “In order to evaluate the validity of proposed explanations, students must develop enough confidence in their reasoning abilities to question others’ mathematical arguments as well as their own” (NCTM, 2000, 345-346). Thus, PSSM aligns the expectation for students to engage in logical reasoning with the work of doing mathematics, getting closer to proponents of the mathematical argument in this regard.

The 1989 Standards had reflected some of the goals of the formal argument within the context of integration of technology. The 1989 Standards had mentioned computer software as useful to “develop, compare, and apply algorithms” (p. 159). In PSSM, however, the discussion on how to use computers to experiment the making of deductive arguments seem to be more aligned with a mathematical argument. The evolution of dynamic geometry software to include less aspects of programming and more aspects of manipulation appears to have affected the goals of teaching geometry. Earlier kinds of educational software required some basic programming skills for which the learning of formal logic was a resource; current dynamic geometry software tend to demand uses of logic more tied to the semantics of the domain being studied.

A strong reminiscence of the humanist discourse of the Committee of Ten, and of the mental discipline philosophy that influenced them, is the notion that all students should learn geometry, implied in the NCTM motto of “mathematics for all.” This is hardly news for NCTM, which was founded to counter some educational impetus to make mathematics optional (Betz, 1936; Kilpatrick et al., 1920). PSSM does not establish different goals for students depending upon their ability, similar to the report of the Committee of Fifteen.

Students’ work in crafting logical arguments is aligned with the work of mathematicians, making the Geometry Standard a case of the mathematical argument. For example, PSSM offers a vignette in which students fail to find a generalization as they look at polygons that result from connecting the midpoints of the sides of different polygons. The teacher in the vignette emphasizes the value of the process students were engaged with and commends students for engaging in a process that “is truly mathematical” (NCTM, 2000, p. 312).

While the Geometry Standard draws no support from the formal argument, the influence of the utilitarian argument is perhaps as salient as that of the mathematical argument. The Geometry Standard shows applications of geometric notions in the workplace. “Applied problems can furnish both rich contexts for using geometric ideas and practice in modeling and problem solving” (NCTM,
2000, p. 313). The mathematics of the engineers designing a pipeline, of the artist using perspective drawing, of the worker finding the minimum path, and of the banker setting the best route is the mathematics of students’ future careers.

The choice of different geometries in PSSM is sustained by the need to solve real-world problems as within proponents of the utilitarian argument (NCTM, 2000, p. 316-317). The authors of PSSM state that “in one set of circumstances it might be most useful to think about an object’s properties from the perspective of Euclidean geometry whereas in other circumstances, a coordinate or transformational approach might be more useful” (NCTM, 2000, p. 309). Thus, whereas the 1989 Standards justified allusion to non-Euclidean geometries on the need to illustrate the role of assumptions in an axiomatic system, the choice of geometries other than synthetic Euclidean geometry in PSSM is more aligned with the utilitarian notion of finding the best tool to solve a problem.

Traces of the intuitive argument also appear in the Geometry Standard. Students are to develop visual, spatial, and drawing skills as well as a language to communicate about their experiences in an increasingly human-made, geometry based material world (Tatsuoka et al., 2004). Computers play a special role to develop their geometric intuition. For example, the Geometry Standard emphasizes the use of computer software to develop students’ visualization skills needed in job settings (NCTM, 2000, p. 316). There is the sense that geometry will help in having a better understanding of the world as Usiskin (1980/1995) had argued.

The Geometry Standard follows the trend started in the report of the Committee of Fifteen, where the fusion between geometric concepts and applications became relevant in the geometry curriculum. At the same time, the Geometry Standard incorporates some of expectations for the geometry course developed during the 20th century such as having students experience the work of mathematicians, preparing them for their future careers, and developing skills unique to the study of geometry. While the Geometry Standard has acknowledged those goals for the study of geometry, it has abandoned the notion of transfer of training in logical reasoning that characterized earlier debates about the study of geometry.

The articulation of competing arguments within the Geometry Standard might reflect tensions about the goals of schooling. Different justifications might be in conflict with each other and at the same time might help in supporting diverse versions of the geometry course.

Competing visions—that is, competing answers to the questions of what we should teach, why we should teach one thing rather than another, and who should have access to what knowledge—can be healthy, but only if they are recognized and dealt with. It is naive, moreover, to assume that wide-ranging reform in school mathematics will result from any effort that focuses only on schools and is not somehow linked to reform of the wider society. (Stanic & Kilpatrick, 1992, p. 416)
The 21st century may also bring about new justifications for the geometry course, especially with the availability of new technology and other resources and with the advent of more stringent accountability demands for teachers, schools, and districts.

**Conclusion**

In this article we have described four arguments that have been used during the 20th century to justify the high school geometry course in the US. At the onset of the 20th century, geometry was justified on the grounds of a *formal argument*—that geometry helped discipline the mental faculties of logical reasoning. At various times during the 20th century other arguments emerged recurrently. A *utilitarian argument* was an incipient influence in the report of the Committee of Fifteen, which recommended the teaching of applications of geometry. The argument was that geometry would provide tools for students’ future work or non-mathematical studies. A *mathematical argument* was also an incipient influence in the report of the Committee of Fifteen, but came to the fore with more force at about mid century, justifying the geometry course as an opportunity for students to experience the work and ideas of mathematicians. The *mathematical argument* recommended the study of geometry because of its capacity to engage students in making and proving conjectures or to illustrate for students how dramatic conceptual developments occur in the discipline of mathematics that permit to solve a multitude of new problems. Finally, an *intuitive argument* emphasized the role of geometry providing students with an interface language and a representation system to relate to the real world.

The value of the distinction between different arguments is apparent as we look at the argument made in *PSSM* for the study of geometry at the end of the 20th century: This argument draws on a combination of the modal arguments offered during the 20th century but has a distinctive flavor, quite different than the mental disciplinarian call for the study of geometry at the end of the 19th century. The four modal arguments can be used to describe specific curriculum approaches and note what is at stake in the geometry instruction of specific institutions. Whatever an institution puts at stake in a course of studies, interaction in classrooms develops around the procurement of those stakes, even if the mathematics “constituted through teaching” (Høyrup, 1994, p. 3) does not reduce to the claiming of those stakes. The four modal arguments thus identify expectations that might shape what a teacher and her class at work are pursuing—what they hold themselves accountable for vis-à-vis the subject of studies.

The 20th century showcases the history of the rise and fall of the formal argument as the main reason for students to study geometry. Our work shows how three other arguments emerged through the century to justify the course, adding conditions and constraints to the work that teachers and students do in classrooms. One could therefore expect that the contents of the geometry course of any institution have become more heterogeneous as these various arguments have come to the fore. But also this heterogeneity has contributed to sustain the
geometry course in spite of the gradual fall of the formal argument. *PSSM* exemplifies how the other three arguments can integrate to justify the study of geometry, if not the geometry course.

But those developments in the justification for the geometry course are the discussions at the level of the opinion leaders and policy makers. Actual schools, parents, teachers, and students might well continue to hold geometry instruction accountable to procure the stakes identified by the formal argument. Our research suggests that at a minimum, instructional policy that seeks to promote the vision of *PSSM* will have to contend with those expectations and find a serious way to talk to stakeholders about the kind of transfer that is reasonable to expect from school studies.

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**Notes**

1. The order of authorship is alphabetical.

2. For sake of brevity we do not include here the analysis of textbooks and will present them on a separate piece.

3. It is possible for example to think that the record in *School Science and Mathematics, School Review, the American Mathematical Monthly*, or *L’Enseignement Mathématique* (if we wanted to think globally) would yield complementary categories.

4. Slaught would later be named honorary president of NCTM from 1936-1938.

5. Cajori was the only member of the Committee of Fifteen who also participated as a member of the Mathematics Commission that reported to the Committee of Ten.

6. Halbert Christofferson was a professor of mathematics in the Mathematics Department and director of the Secondary Education program at the School of Education at Miami University, Ohio. He wrote a book for geometry teachers based upon his dissertation work at Columbia University. He founded with other teachers, including Harold Fawcett, the Ohio Council of Mathematics Teachers.

7. E. R. Breslich presided the School Science and Mathematics Association between 1926 and 1927 and NCTM between 1939 and 1941.
8. Later a professor of education at the University of Illinois, Henderson became a pioneer in research on mathematics teaching.

9. Moise had studied mathematics with the celebrated topologist R. L. Moore who had advocated for a similar experiential pedagogy in the training of professional mathematicians (see Burton-Jones, 1977; Zitarelli, 2006).

10. The other content strands are number and operations, algebra, measurement, and data analysis and probability.

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